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Non-standard Quantum Groups and Superization

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We obtain the universal R-matrix of the non-standard quantum group associated to the Alexander-Conway knot polynomial. We show further that this non-standard quantum group is related to the super-quantum group $U_q gl(1|1)$ by a general process of superization, which we describe. We also study a twisted variant of this non-standard quantum group and obtain, as a result, a twisted version of $U_q gl(1|1)$ as a q-supersymmetry of the exterior differential calculus of any quantum plane of Hecke type, acting by mixing the bosonic x_i co-ordinates and the forms $\mathrm{d}x_i$.

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I Introduction

Standard quantum group deformations $U_q g$ have been studied extensively in recent years as deformations of the enveloping algebras of ordinary Lie algebras [1] [2]. One has techniques for their construction coming from quantum and classical inverse scattering (such as the FRT description [3] and the quantum double [1] of Drinfeld which provides their universal R-matrix or quasitriangular structure). One also has the possibility of geometrical applications in analogy with the role of the corresponding Lie algebras in the undeformed case.

Quite mysterious however, remain the quantum groups which one can build in association to other, non-standard, solutions R of the quantum Yang-Baxter equations (QYBE) $R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}$. The solutions of these equations form a variety which has an interesting structure [4] containing much more than the standard solutions and their twistings. In low dimensions its structure consists [5] of several families, and the corresponding matrix bialgebras $A(R)$ in these low-dimensional cases are computed [6] but remain little understood. For example, existence of a dual-quasitriangular structure for general $A(R)$ is known from [7], hence existence, at least formally, of a universal R-matrix for the corresponding 'enveloping algebra' dual to it is also known, but there is no easy algorithm to compute it (in general there may be nothing like a borel subalgebra and associated quantum double as there is in the standard case). The physical significance of such non-standard quantum groups is likewise not at all understood because of the absence of the analogies with undeformed objects which are possible in the standard case.

In this paper, we study the simplest and most well-known of these non-standard quantum groups in some detail. As a Hopf algebra, it was studied in [8] but its full structure, notably the universal R-matrix of its enveloping-type algebra $U_q(H_1, H_2, X^{\pm})$, has not been found before the announcement of the present paper in [9]. We begin in Section II by giving the full derivation of this, which is by directly solving the non-linear equations for the quasitriangular structure. It is, to our knowledge, the first non-standard quantum group with non-trivial universal R-matrix known explicitly. Many of the standard constructions for quantum groups depend critically on the

universal R-matrix and our result means that these now hold automatically for this non-standard quantum group. These include the tensor product of tensor operators for the quantum group [7], quantum traces, and link and 3-manifold invariants. The link invariants here depend on the ribbon element, which we compute, and traces in irreducible representations. The matrix R underlying this example is known to be connected with the Alexander-Conway knot polynomial [10], which is usually developed in connection with free fermions and the super-quantum group $U_q gl(1|1)$ [11] [12], as well as in connection with other quantum algebras [13].

In Section III, we make fully precise the connection suggested here between this non-standard quantum group and $U_q gl(1|1)$, through a process which we call *superization*. This is the second goal of the paper. This depends on a general algebraic principle of transmutation introduced (by the first author) in [14] as a way to shift the category (in this case bosonic or super) in which an algebraic structure lives. We show that this procedure agrees, for certain R-matrices, with a more ad-hoc process of superization in which [15] certain types of solutions of the QYBE can be converted to solutions of the super QYBE (SYBE). We see this for our example. This elucidation is important because it tells us that certain families of non-standard quantum groups, corresponding to certain curves in the Yang-Baxter variety of solutions, correspond after systematic superization to q-deformations of super-Lie algebras. Without superization, we would have to view super-Lie algebras as some independent theory 'analogous' to the standard theory of $U_q g$, but we see now that this theory is precisely equivalent to the theory of certain non-standard bosonic quantum groups in the same framework as their more standard cousins. One can take this further and look for other more novel algebraic objects corresponding to still other families of non-standard quantum groups by further transmutation procedures. A step in this latter direction is the sequel to the present paper [16].

In Section IV, we show how transmutation works for the corresponding quantum groups of function algebra type. Again, there is a general theory and a matching of our algebraic constructions with the more ad-hoc process of replacing certain R-matrices by super versions. We demonstrate the construction on the non-standard quantum function algebra [8] corresponding to the Alexander-

Conway R-matrix as in the previous section. This time, superization gives the super-quantum group $GL_q(1|1)$ as studied for example in [17]. Thus, once again, superization systematically relates two different streams in the literature. In the present case we obtain, as an example, the (non-central) q-determinant for the non-standard quantum group, corresponding to the known super q-determinant of $GL_q(1|1)$. The same relation holds between certain non-standard quantum groups and $GL_q(n|m)$, among other examples.

In Section V, we proceed to a new application of (a twisting of) our particular algebra $U_q(H_1, H_2, X^{\pm})$ and its superization, namely to q-deformed geometry. A great many papers have been devoted in recent years to the study of the exterior algebra Ω_q of non-commuting 'co-ordinates' x_i and differential forms $\mathrm{d}x_i$ [18] [19] on quantum planes. We show that, at least in the case when the exchange relations governing the quantum plane are of Hecke type (e.g. the standard su_n family of quantum planes, but others as well), the exterior algebra of differential forms has as a novel q-symmetry a non-standard quantum group $U_q^{\Omega}(H_1, H_2, X^{\pm})$ obtained by twisting $U_q(H_1, H_2, X^{\pm})$ by a quantum cocycle in the sense of [20] [21]. The systematic theory of superization tells us, further, that this corresponds equivalently to a q-supersymmetry $U_q^{\Omega} gl(1|1)$ obtained by a super-cocycle twist of $U_q gl(1|1)$. This is a general result which works for all Hecke q-spaces of any dimension, with the odd generators acting by mixing the x_i and the $\mathrm{d}x_i$. As such, it generalizes an important classical supersymmetry of the exterior algebra of differential forms which has been used (for example) in [22] and [23], where it figures in a fundamental way. This q-deformed $gl(1|1)$ supersymmetry of q-differential calculi does not appear to be discussed elsewhere, and is a useful outcome of our superisation techniques. It may also be possible to obtain its matrix superalgebra form as a quotient of a general 'universal superbialgebra' construction [24], which has been pointed out to us.

This paper is the final version of a 1991 preprint with similar title. We have added Section V with the new application to q-geometry. The earlier sections appear to us to remain of interest, as well as playing a role in [25] and other subsequent works.

II Quasitriangular Structure on $U_q(H_1, H_2, X^\pm)$

This section studies the Hopf algebra of quantum enveloping algebra type associated to the Alexander-Conway matrix solution

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \tag{1}$$

of the QYBE. The main result is an explicit expression of its universal R-matrix $\mathcal{R}$ or quasitriangular structure obeying the axioms of Drinfeld [1]. From this we show further that the Hopf algebra is ribbon as well. We also compute $\mathcal{R}$ in all finite irreducible representations of the algebra at generic q.

We begin by constructing the quantum 'enveloping' algebra by analogy with the FRT approach [3]. I.e. we consider two matrices of generators $\mathbf{l}^\pm$ and quadratic relations $R\,\mathbf{l}_2^\pm\,\mathbf{l}_1^\pm = \mathbf{l}_1^\pm\,\mathbf{l}_2^\pm\,R$, $R\,\mathbf{l}_2^+\,\mathbf{l}_1^- = \mathbf{l}_1^-\,\mathbf{l}_2^+\,R$ in the standard notation $\mathbf{l}_1^\pm = \mathbf{l}^\pm \otimes \mathrm{id}$, $\mathbf{l}_2^\pm = \mathrm{id} \otimes \mathbf{l}^\pm$. This forms a bialgebra $\widetilde{U(R)}$ with $\triangle \mathbf{l}^\pm = \mathbf{l}^\pm \otimes \mathbf{l}^\pm$, $\varepsilon(\mathbf{l}^\pm) = \mathrm{id}$ as usual. We now quotient by adding further relations, or equivalently, by making an ansatz for the specific form of $\mathbf{l}^\pm$. There is no algorithm for this (apart from some general remarks in [7]) but the lower and upper triangular ansatz

$$\mathbf{l}^+ = \begin{pmatrix} K_1 & 0 \\ (q-q^{-1})\,X^+ & K_2^{-1} \end{pmatrix}, \qquad \mathbf{l}^- = \begin{pmatrix} K_1^{-1} & -\,(q-q^{-1})\,X^- \\ 0 & K_2 \end{pmatrix} \tag{2}$$

works and gives as quotient of $\widetilde{U(R)}$ the Hopf algebra $U_q(K_1, K_2, X^\pm)$ say, studied in [8] as generated by 1, K_i, K_i^{-1}, $X^\pm$, $i = 1, 2$ and relations

$$\begin{aligned} &K_i \cdot K_i^{-1} = K_i^{-1} \cdot K_i = 1, \qquad [K_1, K_2] = 0, \\ &K_1 X^\pm = q^{\pm 1} X^\pm K_1, \qquad K_2 X^\pm = -q^{\mp 1} X^\pm K_2, \\ &[X^+, X^-] = \frac{K_1 K_2 - K_1^{-1} K_2^{-1}}{q - q^{-1}}, \qquad (X^\pm)^2 = 0. \end{aligned} \tag{3}$$

Here q is an arbitrary parameter. The coalgebra structure is given by

$$\triangle K_i = K_i \otimes K_i, \qquad \triangle K_i^{-1} = K_i^{-1} \otimes K_i^{-1},$$

$$\triangle X^{+} = X^{+} \otimes K_1 + K_2^{-1} \otimes X^{+}, \qquad \triangle X^{-} = X^{-} \otimes K_2 + K_1^{-1} \otimes X^{-}, \tag{4}$$
$$\varepsilon(K_i) = \varepsilon\left(K_i^{-1}\right) = 1, \qquad \varepsilon(X^{\pm}) = 0$$

and the antipode S by

$$\begin{gathered} S(K_i) = K_i^{-1}, \qquad S\left(K_i^{-1}\right) = K_i, \\ S(X^{+}) = -q\, K_1^{-1} K_2 X^{+}, \qquad S(X^{-}) = q\, K_1 K_2^{-1} X^{-}. \end{gathered} \tag{5}$$

This is essentially the Hopf algebra which we study in this section. Notice that the relations $(X^{\pm})^2 = 0$ are highly suggestive of a superalgebra with $X^{\pm}$ odd and K_1, K_2 even elements. Yet this is an ordinary Hopf algebra and not at all a super-quantum group because to extend the algebra homomorphism $\triangle$ consistently to products of generators is necessary to use the bosonic manipulation $(a \otimes b)(c \otimes d) = a\, c \otimes b\, d$ for the tensor product of two copies. In the super-quantum group case we would need $(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\ \deg(c)} a\, c \otimes b\, d$, which turns out to be inconsistent with the relations (3). In other words, this non-standard quantum group reminds us of a super-quantum group but is an ordinary bosonic one. This is a puzzle that we address in Section III.

We now introduce new generators for the above non-standard quantum 'enveloping algebra', namely

$$K_1 = q^{H_1/2}, \qquad K_2 = e^{i\frac{\pi}{2} H_2}\, q^{H_2/2}. \tag{6}$$

Definition II.1 *We denote by* $U_q(H_1, H_2, X^{\pm})$ *the above non-standard Hopf algebra with these new generators* $\{H_1, H_2, X^{\pm}\}$ *and the corresponding relations*

$$\begin{gathered} [H_1, H_2] = 0, \qquad [H_1, X^{\pm}] = \pm 2 X^{\pm}, \qquad [H_2, X^{\pm}] = \mp 2 X^{\pm}, \\ [X^{+}, X^{-}] = \frac{K_1 K_2 - K_1^{-1} K_2^{-1}}{q - q^{-1}}, \qquad (X^{\pm})^2 = 0, \end{gathered} \tag{7}$$

The coalgebra and antipode for the new generators is $\triangle H_i = H_i \otimes 1 + 1 \otimes H_i$ *and* $\varepsilon(H_i) = 0$ $(i = 1, 2)$ $S(H_i) = -H_i$ *(unchanged for the rest).*

This change to new primitive generators H_i is familiar for the standard quantum groups $U_q g$ and can be made precise using formal power series in the deformation parameter [1]. The novel feature in our non-standard case, however, is the additional factor $e^{i\pi H_2/2}$ in (6) which requires more work to make precise. We proceed formally but note that such exponentials do have a well-defined meaning for the operator representations of H_2 which we consider. An alternative is to adjoin $g = e^{i\pi H_2/2}$ as a seperate group-like element, with corresponding commutation relations.

We are now ready to obtain the quasitriangular structure for this new Hopf algebra. Recall that a Hopf algebra U is called *quasitriangular* if there is an invertible element $\mathcal{R}$ in $U \otimes U$ that obeys the axioms [1]

$$\triangle' (a) = \mathcal{R} \triangle (a) \mathcal{R}^{-1} \tag{8}$$

for all a in U and

$$(\triangle \otimes \mathrm{id}) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}, \qquad (\mathrm{id} \otimes \triangle) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}. \tag{9}$$

Here $\triangle'$ is defined as $\tau \circ \triangle$ where $\tau(x \otimes y) \mapsto y \otimes x$. The comultiplication $\triangle'$ gives a second Hopf algebra structure on U in addition to $\triangle$ and the '*universal R-matrix*' $\mathcal{R}$ intertwines them. Equations (9) are evaluated in $U^{\otimes^3}$ and when $\mathcal{R}$ is expressed as the formal sum $\mathcal{R} = \sum_i a_i \otimes b_i$, $\mathcal{R}_{12}, \mathcal{R}_{13}$ and $\mathcal{R}_{23}$ are then given by the standard notation $\mathcal{R}_{12} = \sum_i a_i \otimes b_i \otimes 1$, $\mathcal{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i$ and $\mathcal{R}_{23} = \sum_i 1 \otimes a_i \otimes b_i$. Then $\mathcal{R}$ satisfies quantum Yang-Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \tag{10}$$

in $U^{\otimes 3}$ as one sees easily from (8) and (9).

Theorem II.2 *The Hopf algebra $U_q(H_1, H_2, X^{\pm})$ is quasitriangular with universal R-matrix*

$$\mathcal{R} = e^{-i\frac{\pi}{4} H_2 \otimes H_2} q^{\frac{1}{4}(H_1 \otimes H_1 - H_2 \otimes H_2)} \Big(1 \otimes 1 + (1 - q^2) E \otimes F\Big), \tag{11}$$

where E and F denote the elements $K_2 X^+$ and $K_2^{-1} X^-$ respectively.

Proof We start proving Drinfeld's axioms (8). For brevity we shall use the shorthand notation $q^{H\otimes H}$ to denote the combination $e^{-i\frac{\pi}{4}H_2\otimes H_2}\, q^{\frac{1}{4}(H_1\otimes H_1 - H_2\otimes H_2)}$ in $\mathcal{R}$ above. It is clear that (8) is satisfied for each $a = H_i$ since $\triangle' H_i \,(= \triangle H_i)$ commutes with $q^{H\otimes H}$ and $E\otimes F$. It is also satisfied for $a = E$ because $(K_1K_2\otimes E + E\otimes 1)\, q^{H\otimes H} = q^{H\otimes H}\left(1\otimes E + E\otimes K_1^{-1}K_2^{-1}\right)$ and $\left(1\otimes E + E\otimes K_1^{-1}K_2^{-1}\right)(1\otimes 1 + (1-q^2)E\otimes F) = (1\otimes 1 + (1-q^2)E\otimes F)\,(1\otimes E + E\otimes K_1K_2)$, which put together produce $\triangle' E\cdot\mathcal{R} = (K_1K_2\otimes E + E\otimes 1)\, q^{H\otimes H}\,(1\otimes 1 + (1-q^2)E\otimes F) = \mathcal{R}\cdot\triangle E$. In an analogous manner we see that the relation is satisfied for $a = F$ as well so we conclude that (8) holds for the entire Hopf algebra.

We prove now the first half of (9). From $(\triangle\otimes \mathrm{id})\; H_i\otimes H_i = H_i\otimes 1\otimes H_i + 1\otimes H_i\otimes H_i$ we have that $(\triangle\otimes\mathrm{id})\; q^{H\otimes H} = q^{H\otimes 1\otimes H}\, q^{1\otimes H\otimes H}$, with $q^{H\otimes 1\otimes H}$ and $q^{1\otimes H\otimes H}$ here understood in obvious notation. In turn

$$(\triangle\otimes\mathrm{id})\left(1\otimes 1 + (1-q^2)\,E\otimes F\right) = \left(1^{\otimes^3} + (1-q^2)\,E\otimes K_1K_2\otimes F\right)\left(1^{\otimes^3} + (1-q^2)\,1\otimes E\otimes F\right)$$

which gives

$$(\triangle\otimes\mathrm{id})\,\mathcal{R} = q^{H\otimes 1\otimes H}\left(1^{\otimes^3} + (1-q^2)\,E\otimes 1\otimes F\right) q^{1\otimes H\otimes H}\left(1^{\otimes^3} + (1-q^2)\,1\otimes E\otimes F\right) = \mathcal{R}_{13}\mathcal{R}_{23},$$

once the commutator $q^{1\otimes H\otimes H}\,(E\otimes K_1K_2\otimes F) = (E\otimes 1\otimes F)\, q^{1\otimes H\otimes H}$ is used. The remaining equation in (9) is proved with a similar procedure. $\square$

This goes significantly beyond the scope of ref. [8], where the quasitriangular structure $\mathcal{R}$ is not investigated. It is crucial for numerous applications. In particular, the quantities $u = \sum_i S(b_i)\, a_i$, $S(u)$, $z = u\,S(u) = S(u)\,u$ and $r = u\,(S(u))^{-1} = (S(u))^{-1}\,u$ are of special interest in the general theory [26]. Here the invertible element u implements the square of the antipode in the form $uau^{-1} = S^2(a)$ for all elements a of the quantum group, z is necessarily central because it commutes with any a, and r is group-like, i.e. $\triangle r = r\otimes r$, and satisfies $rar^{-1} = S^4(a)$. We have,

Lemma II.3 *The elements* u, $S(u)$, z, r *in* $U_q\left(H_1, H_2, X^{\pm}\right)$ *are given by*

$$u = e^{i\frac{\pi}{4}H_2^2} q^{-\frac{1}{4}\left(H_1^2 - H_2^2\right)} \Big(1 + (1-q^2)\, K_1^{-1} K_2^{-1} F E\Big)\,, \tag{12}$$

$$S(u) = e^{i\frac{\pi}{4}H_2^2} q^{-\frac{1}{4}\left(H_1^2 - H_2^2\right)} \Big(1 + (1-q^2)\, E F K_1 K_2\Big)\,, \tag{13}$$

$$z = e^{i\frac{\pi}{2}H_2^2} q^{-\frac{1}{2}\left(H_1^2 - H_2^2\right)} \Big(1 + (1-q^2) \Big(K_1 K_2 E F + K_1^{-1} K_2^{-1} F E\Big)\Big)\,, \tag{14}$$

$$r = e^{-i\pi H_2} q^{-(H_1+H_2)}. \tag{15}$$

Proof First we evaluate the element u. The universal R-matrix (11) can be written in a better way for the present calculations as the serie $\sum_{l,n=0}^{\infty} \frac{(-1)^n}{l!n!} \left(\frac{1}{4}\ln q\right)^l \left(i\frac{\pi}{4} + \frac{1}{4}\ln q\right)^n \left(H_1^l H_2^n \otimes H_1^l H_2^n\right)$ $\left(1^{\otimes^2} + (1-q^2) E \otimes F\right)$. To the sum $\sum_i S(b_i)\, a_i$ contribute the terms $S\left(H_1^l H_2^n\right) H_1^l H_2^n$ and $S\left(H_1^l H_2^n F\right) H_1^l H_2^n E$, equal to $(-1)^{l+n} H_1^{2l} H_2^{2n}$ and $(-1)^{l+n+1} K_1 K_2 \left(H_1+2\right)^{2l} \left(H_2-2\right)^{2n} F E$, respectively. Restoring from these contributions the unexpanded form of u what we obtain is the expression (12). Relation (13) is the result of acting with the antipode map on (12) and taking into account that S is an antialgebra homorphism so that $S(ab) = S(b)\, S(a)$. The element z in (14) is the product of results (12) and (13). Concerning the derivation of (15) it requires the inverse of $S(u)$ given by

$$\left(S(u)\right)^{-1} = e^{-i\frac{\pi}{4}H_2^2} q^{\frac{1}{4}\left(H_1^2 - H_2^2\right)} \Big(1 - (1-q^2)\, E F K_1^{-1} K_2^{-1}\Big)$$

as one can check easily. The final expression in (15) is calculated then from (12) and this previous result. We complete in this way the proof of the lemma. □

The elements $\mathcal{R}$ and u, etc., of a quasitriangular Hopf algebra are the key to numerous applications of quantum groups, among them the construction of link and 3-manifold invariants. In this context, a quasitriangular Hopf algebra is called *ribbon* if there is a central element ν in it such that ν satisfies [27]

$$\nu^2 = u\, S(u), \quad \triangle\nu = \left(\mathcal{R}_{21}\mathcal{R}\right)^{-1} (\nu \otimes \nu)\,, \quad \varepsilon(\nu) = 1, \quad S(\nu) = \nu,$$

where $\mathcal{R}_{21}$ denotes $\tau \circ \mathcal{R} = \sum_i b_i \otimes a_i$.

Proposition II.4 *$U_q(H_1, H_2, X^\pm)$ is a ribbon Hopf algebra. The element ν and its inverse are given by*

$$\begin{aligned} \nu &= e^{i\frac{\pi}{4}H_2^2} q^{-\frac{1}{4}\left(H_1^2 - H_2^2\right)} K_1 K_2 \left(1 + (1-q^2) K_1^{-1} K_2^{-1} F E\right) \\ \nu^{-1} &= e^{-i\frac{\pi}{4}H_2^2} q^{\frac{1}{4}\left(H_1^2 - H_2^2\right)} K_1^{-1} K_2^{-1} \left(1 - (1-q^2)\, K_1 K_2 F E\right). \end{aligned} \tag{16}$$

Proof It is easy to see that indeed $\nu^2 = z$ given in (14). That $\triangle\nu = (\mathcal{R}_{21}\mathcal{R})^{-1}(\nu \otimes \nu)$ can be checked with (11) and (16). The remaining two properties derive from (4) and (5) in a straightforward manner. Finally, that $\nu \cdot \nu^{-1} = \nu^{-1} \cdot \nu = 1$ is again a simple computation. $\square$

About the general properties of $U_q(H_1, H_2, X^\pm)$ we mention that other elements of interest are the casimirs (central elements) given by

$$c_1^2 = e^{i\pi H_2} q^{H_1 + H_2}, \qquad c_2 = X^+ X^- - \frac{1}{2}\left(\frac{K_1 K_2 - K_1^{-1} K_2^{-1}}{q - q^{-1}}\right).$$

Notice that $r = c_1^{-2}$ so r too is central in our case. This indicates that the antipode has order 4, that is to say $S^4 = \mathrm{id}$. The element $c_1 \equiv e^{i\frac{\pi}{2}H_2} q^{(H_1+H_2)/2}$ is not central but anticommutes with $X^\pm$.

We turn now to one of the original applications of the universal R-matrix, namely to obtain matrix solutions of the QYBE. Since $\mathcal{R}$ obeys (10) abstractly in the algebra, so does the matrix of $\mathcal{R}$ in any representation of the quasitriangular Hopf algebra. In the R-matrix setting [3] we know that the bialgebra $\widetilde{U(R)}$ is dually paired with the quantum matrix bialgebra $A(R)$, which leads to the *canonical representation* ρ defined by [3] [7]

$$\rho^i{}_j\,(l^{+k}{}_l) = R^i{}_j{}^k{}_l, \qquad \rho^i{}_j\,(l^{-k}{}_l) = R^{-1}\,{}^k{}_l{}^i{}_j, \tag{17}$$

in terms of our R-matrix. In the case of the AC solution (1) this gives us a representation

$$\begin{aligned} \rho(K_1) &= \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho(K_2) = \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}, \\ \rho(X^+) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \rho(X^-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{18}$$

Finite dimensional irreducible representations of the algebra $U_q(K_1, K_2, X^\pm)$ have been studied in [8], where it is shown that the only 'faithful' ones among them are two-dimensional and equivalent to the following

$$\begin{gathered}
\pi(K_1) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & q^{-1}\lambda_1 \end{pmatrix}, \qquad \pi(K_2) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & -q\lambda_2 \end{pmatrix}, \\
\pi(X^+) = \begin{pmatrix} 0 & \left(\lambda_1\lambda_2 - \lambda_1^{-1}\lambda_2^{-1}\right)/\left(q - q^{-1}\right) \\ 0 & 0 \end{pmatrix}, \qquad \pi(X^-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\end{gathered} \tag{19}$$

where λ_1, λ_2 are any non zero complex constants such that $(\lambda_1\lambda_2)^2 \neq 1$. Here a representation is said 'faithful' in the spirit of [8] if $\pi(a) = \pi(b)$ implies $a = b$ for a, b in the vector space spanned by $\{K_i, K_i^{-1}, X^\pm\}$. Each representation (19) is labelled by two numbers (λ_1, λ_2) and the tensor product of two of them decomposes in irreducible representations as the following law indicates

$$(\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2) = (\lambda_1\mu_1, \lambda_2\mu_2) \oplus \left(q^{-1}\lambda_1\mu_1, -q\lambda_2\mu_2\right).$$

We use these irreducible representations for $U_q\left(H_1, H_2, X^\pm\right)$ as well, parametrizing them as

$$\lambda_1 = q^{m_1}, \qquad \lambda_2 = e^{i\pi m_2} q^{m_2},$$

where m_1, m_2 are such that $q^{2(m_1+m_2)} e^{2\,i\pi m_2} \neq 1$. The corresponding representation $[m_1, m_2]$ is

$$\pi(H_1) = \begin{pmatrix} 2m_1 & 0 \\ 0 & 2\left(m_1 - 1\right) \end{pmatrix}, \qquad \pi(H_2) = \begin{pmatrix} 2m_2 & 0 \\ 0 & 2\left(m_2 + 1\right) \end{pmatrix}, \tag{20}$$

together with $\pi(X^\pm)$ as in (19).

For these representations we can state the following proposition

Proposition II.5 *In the canonical representation $\rho = [1, 0]$ we recover $\rho \otimes \rho\,(\mathcal{R}) = R$, the AC solution (1) of Section II. In the general representation $\pi = [m_1, m_2]$ of $U_q\left(H_1, H_2, X^\pm\right)$ we have that $\pi \otimes \pi\,(\mathcal{R})$ is again (1) with the substitution of q by $e^{i\pi m_2} q^{m_1+m_2}$.*

Proof This follows by direct computation. $\square$

In the above proposition, the part concerning ρ is to be expected from the general theory of matrix quantum groups (see [7], Sec. 3) which therefore serves as a check on our universal R-matrix (11). With respect to $\pi \otimes \pi (\mathcal{R})$ we see that it does not provide new or unfamiliar solutions to the QYBE but a reparametrization of the AC solution again, cf. remarks in [8] for other aspects of these representations. On the other hand, there are certainly other representations of $U_q (H_1, H_2, X^{\pm})$ (for example the infinite dimensional left-regular one) on which our universal $\mathcal{R}$ can provide new braid group actions and corresponding invariants.

III Connection of $U_q (H_1, H_2, X^{\pm})$ with super $U_q gl (1|1)$

In the last section we pointed out that the relations $(X^{\pm})^2 = 0$ are indicative of some kind of super-Hopf algebra structure. Yet $U_q (H_1, H_2, X^{\pm})$ is an ordinary Hopf algebra and therefore not a super one at all. This puzzle was raised in [8] and we give now some insight into it by means of the transmutation theory of [28] [14], which enunciates that under suitable circumstances is possible to transform Hopf algebras into super-Hopf algebras and vice-versa. As an application of this superization construction we prove here that the superization of (a quotient of) $U_q (H_1, H_2, X^{\pm})$ coincides with the super-quantum group $U_q gl (1|1)$. We also study this procedure more generally and connect it with a procedure of superizing the R-matrix itself under certain conditions.

We start with an algebraic *superization* procedure for ordinary Hopf algebras as follows. It is a special case of a theory in [14].

Proposition III.1 *If H is a Hopf algebra containing a group-like element g such that $g^2 = 1$, there is a super-Hopf algebra $\underline{H}$, its superization, defined as the same algebra and counit as H, and the comultiplication, antipode (if any) and quasitriangular structure (if any) of H modified to*

$$\underline{\triangle} h = \sum h_{(1)} \, g^{-\deg (h_{(2)})} \otimes h_{(2)}, \qquad \underline{S} (h) = g^{\deg (h)} \, S (h) \tag{21}$$

and

$$\underline{\mathcal{R}} = \mathcal{R}_g^{-1} \sum_i a_i \, g^{-\deg\,(b_i)} \otimes b_i. \tag{22}$$

Here h denotes an arbitrary element of H with comultiplication $\triangle h = \sum h_{(1)} \otimes h_{(2)}$ in the standard notation and $\mathcal{R} = \sum_i a_i \otimes b_i$ denotes the universal R-matrix of H. The element $\mathcal{R}_g = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) = \mathcal{R}_g^{-1}$ is the nontrivial universal R-matrix of the Hopf algebra $\boldsymbol{Z}'_2 = \{1, g\}$ with relations $g^2 = 1$, $\triangle g = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g$. When h is regarded as an element of $\underline{H}$ its grading is obtained with the action of g on h in the adjoint representation, that is by $g\,h\,g^{-1} = (-1)^{\deg\,(h)}\,h$ on homogeneous elements. This superization procedure represents a particular case of a general transmutation theory in [14] for a Hopf algebra containing or being mapped from a general quasitriangular Hopf algebra in the role of $\boldsymbol{Z}'_2$. Other examples of transmutation include anyonization [29] [16] and complete transmutation [28].

In order to superize $U_q(H_1, H_2, X^{\pm})$ in this way, we first note that the role of g is played by $e^{i\pi H_2/2}$ introduced in (6). This element $g = e^{i\pi H_2/2}$ has the property that g^2 is central and group-like. Hence it is natural to impose the relation $g^2 = 1$ in the abstract algebra and consider the quotient $U_q(H_1, H_2, X^{\pm})/g^2 - 1$. Moreover, from Section II we know that $\rho(g^2) = 1$ in the induced canonical representation (20), so this further quotient is consistent with the canonical representation and the pairing with $A(R)$. The element g here commutes with H_1, H_2 and anticommutes with $X^{\pm}$. We have

Proposition III.2 *Let $g = e^{i\pi H_2/2}$. The superization of the Hopf algebra $U_q(H_1, H_2, X^{\pm})/g^2 - 1$ is the super-Hopf algebra $U_q gl(1|1)/e^{2\pi i N} - 1$. Here $U_q gl(1|1)$ is defined by generators h, N even and η, η^+ odd and relations*

$$\begin{gathered} [N, \eta] = -\eta, \qquad [N, \eta^+] = \eta^+, \\ \{\eta, \eta^+\} = \frac{q^h - q^{-h}}{q - q^{-1}}, \qquad \eta^2 = 0, \qquad (\eta^+)^2 = 0, \end{gathered} \tag{23}$$

and h central. The supercoalgebra is given by

$$\underline{\triangle}h = h \otimes 1 + 1 \otimes h, \qquad \underline{\triangle}N = N \otimes 1 + 1 \otimes N,$$
$$\underline{\triangle}\eta = \eta \otimes q^{h-N} + q^{-N} \otimes \eta, \qquad \underline{\triangle}\eta^{+} = \eta^{+} \otimes q^{N} + q^{-h+N} \otimes \eta^{+} \tag{24}$$
$$\underline{\varepsilon}(h) = \varepsilon(N) = \varepsilon(\eta) = \varepsilon(\eta^{+}) = 0.$$

This $U_q gl\,(1|1)$ has a super-quasitriangular structure given by the expression

$$\underline{\mathcal{R}} = q^{-(h \otimes N + N \otimes h)} \left(1 \otimes 1 + (1 - q^2)\, q^N\, \eta \otimes q^{-N}\, \eta^{+}\right). \tag{25}$$

Proof We apply the above superization construction to the quotient $U_q\,(H_1, H_2, X^{\pm})\,/g^2 - 1$ which is itself a Hopf algebra since g^2 is central and group-like. The super-quantum group that results then is easily recognizable as $U_q gl\,(1|1)$ given as in (23)-(24) if we redefine the generators as follows

$$h = (H_1 + H_2)\,/2, \qquad N = H_2/2, \qquad \eta = X^{+}, \qquad \eta^{+} = X^{-} g$$

(so that $q^h = q^{(H_1+H_2)/2}$ and $q^N = q^{H_2/2}$). Notice that a direct consequence of this definition is that h is central as stated. Notice also that the supercomultiplication (24) is an algebra homomorphism consistent with relations (23) provided that we use super manipulation. To see it let us compute as an example $\underline{\triangle}\eta^2 = \eta\, q^{-N} \otimes q^{h-N}\, \eta - q^{-N}\, \eta \otimes \eta\, q^{h-N} = 0$. The remaining cases are operated in a similar manner.

The super-quasitriangular structure (25) follows from (11) and the transformation (22), but appears with an overall factor $c \equiv \mathcal{R}_g e^{-i\pi N \otimes N} q^{h \otimes h}$. This factor is central in $U_q gl\,(1|1) \otimes U_q gl\,(1|1)$ and satisfies $(\triangle \otimes \mathrm{id})\, c = c_{13}\, c_{23}$, $(\mathrm{id} \otimes \triangle)\, c = c_{13}\, c_{12}$, so it can be disregarded from the final expression of $\underline{\mathcal{R}}$ without loss of generality. $\square$

Once the super-universal R-matrix is known is possible to find the super-ribbon Hopf algebra structure of $U_q gl\,(1|1)$ whose explicit calculation we omit because it follows the same steps as in the $U_q\,(H_1, H_2, X^{\pm})$ case. Instead we consider another procedure to obtain super-quantum groups based on a super version of the FRT construction. This involves solutions of the super Yang-Baxter

equation (SYBE). It is said that an invertible matrix $\underline{R}$ in $\mathrm{End}\,(V \otimes V)$ is a *super R-matrix* if it obeys the *null-degree condition*

$$\underline{R}^a{}_c{}^b{}_d = 0 \quad \text{when} \quad p(a) + p(b) - p(c) - p(d) \neq 0 \pmod 2, \tag{26}$$

and $\underline{R}$ is a solution of the SYBE [15]

$$(-1)^{p(e)\,(p(f)+p(c))}\, \underline{R}^b{}_e{}^a{}_f\, \underline{R}^i{}_k{}^f{}_c\, \underline{R}^k{}_j{}^e{}_d = (-1)^{p(r)\,(p(s)+p(a))}\, \underline{R}^i{}_p{}^b{}_r\, \underline{R}^p{}_j{}^a{}_s\, \underline{R}^r{}_d{}^s{}_c, \tag{27}$$

where sum over repeated indices is understood. By V we denote a $\boldsymbol{Z}_2$-graded vector space of dimension $d = n + m$ with linear basis $\{e_i\}$, $i = 1, \ldots, d$ and the assumption that all vectors e_i are homogeneous of degree $p(i) \equiv \deg(e_i) = 0$ when $i = 1, \ldots, n$ and degree $p(i) = 1$ when $i = n+1, \ldots, m$. The particularization $p(i) = 0$ for all i in (27) gives the ungraded Yang-Baxter equation of the previous section and makes (26) an empty relation. The condition (26) demanded in the super case is equivalent to taking $\underline{R}$ as an even matrix when the degree of each matrix element is given by the rule $p(\underline{R}^a{}_c{}^b{}_d) = p(a) + p(b) - p(c) - p(d)$. There is no loss of generality in adopting this null degree assumption and, on the contrary, the advantage that is reduces the calculations considerably. Neither of these two conditions is changed by a transformation in V of the type $e'_j = Q_j{}^i e_i$ with $\deg(e'_i) = \deg(e_i)$ for all i, i.e., both relations are invariant under a change of the form $\underline{R} \rightarrow (Q \otimes Q)^{-1}\, \underline{R}\,(Q \otimes Q)$ with Q any non-singular $d \times d$ matrix of null degree.

It is clear that to any super R-matrix is possible to associate a matrix super-bialgebra $\widetilde{U(\underline{R})}$ with generators 1 and $\{m^{\pm i}{}_j\}$ $i, j = 1, \ldots, d$ and algebra and coalgebra relations

$$(-1)^{p(e)(p(c)+p(f))}\, \underline{R}^b{}_e{}^a{}_f\, m^{x\,f}{}_c\, m^{y\,e}{}_d = (-1)^{p(r)(p(a)+p(s))}\, m^{y\,b}{}_r\, m^{x\,a}{}_s\, \underline{R}^r{}_d{}^s{}_c, \tag{28}$$

$$\underline{\triangle} m^{\pm i}{}_j = \sum_{k=1}^{d} m^{\pm i}{}_k \otimes m^{\pm k}{}_j, \quad \underline{\varepsilon}\left(m^{\pm i}{}_j\right) = \delta^i{}_j, \tag{29}$$

where $(x, y) = (+,+), (+,-), (-,-)$. The generators of this bialgebra are defined as homogeneous elements of degree $p\,(m^{\pm\,i}{}_j) \equiv p(i) + p(j) \pmod 2$ and the super-coalgebra structure (29) satisfies the relation (28) provided that again to multiply two copies of the algebra we use the super manipulation $(a \otimes b)\,(c \otimes d) = (-1)^{p(b)\,p(c)}\,(a\,c \otimes b\,d)$. These formulae (28)-(29) are the super version of

the usual formulae on [3], as is clear on writing (28) in the compact form $\underline{R}\,\mathbf{m}_2^{\pm}\,\mathbf{m}_1^{\pm} = \mathbf{m}_1^{\pm}\,\mathbf{m}_2^{\pm}\,\underline{R}$, $\underline{R}\,\mathbf{m}_2^{+}\,\mathbf{m}_1^{-} = \mathbf{m}_1^{-}\,\mathbf{m}_2^{+}\,\underline{R}$. The notation is similar to the ungraded version except that now the tensor product contained in $\mathbf{m}_1^{\pm}$, $\mathbf{m}_2^{\pm}$ is $\boldsymbol{Z}_2$-graded. We consider that the super version of the FRT method consists then in finding suitable anstaze (or additional relations) for the $\mathbf{m}^{\pm}$ in order to obtain a super quantum group as quotient of $\widetilde{U(\underline{R})}$. We consider now how this process is related to the ungraded situation which we have considered before.

Definition III.3 *A solution $R \in M_d \otimes M_d$ of the usual QYBE is called* superizable *if there exists a grading $p(i) \in \{0,1\}$ on the indices such that*

$$R^a{}_c{}^b{}_d = 0 \quad \text{when} \quad p(a) + p(b) - p(c) - p(d) \neq 0 \ (\text{mod}2).$$

It is clear that superizable solutions of the QYBE correspond to super R-matrices via the relation

$$\underline{R}^a{}_c{}^b{}_d = (-1)^{p(a)p(b)}\, R^a{}_c{}^b{}_d \tag{30}$$

We can build a super-bialgebra $\widetilde{U(\underline{R})}$ from this and ask whether or not there is a reasonable ansatz for the super-matrix generators $\mathbf{m}^{\pm}$ so that the resulting super-quantum group matches our algebraic superization procedure starting from an ordinary quantum group built from $\widetilde{U(R)}$.

Proposition III.4 *The AC R-matrix (1) is superizable with $p(1) = 0, p(2) = 1$. The corresponding super R-matrix comes out from (30) as*

$$\underline{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \tag{31}$$

The super-bialgebra $\widetilde{U(\underline{R})}$ has as quotient the quantum group $U_q gl(1|1)$ given by the ansatz

$$\mathbf{m}^{+} = \begin{pmatrix} q^{h-N} & 0 \\ (q - q^{-1})\,\eta & q^{-N} \end{pmatrix}, \qquad \mathbf{m}^{-} = \begin{pmatrix} q^{-h+N} & -\,(q - q^{-1})\,\eta^{+} \\ 0 & q^{N} \end{pmatrix}.$$

Proof We compute $\underline{R}$ from (30) and then insert the stated ansatz into the relations (28)-(29) and recover those for $U_q gl(1|1)$ in Proposition III.2. $\square$

Hence, at least in this case, our abstract superization construction as in Proposition III.1 is compatible with other more ad-hoc ideas of 'superization' consisting of passing from R-matrices to super R-matrices and looking for suitable ansatze on the quadratic FRT bialgebras. It is possible to develop this second method further in such a way as to always match with the algebraic superization. One has to consider p as defining a Hopf algebra homomorphism $\widetilde{U(R)} \to \mathbb{Z}'_2$ as a dual-quasitriangular Hopf algebra. This requires more machinery than we have introduced so far, but is analogous to a slightly different consideration for matrix quantum groups in [30, Appendix].

IV Quantum determinant and antipode for $A(R)$ and its connection with super $GL_q(1|1)$

In this section we look at a non-standard quantum group built from $A(R)$ where R is our solution (1) and connect it by superization with $GL_q(1|1)$ as studied for example in [17]. This is dual to superization in the preceding section but we will see that by adjoining an element with $g^2 = 1$ (rather than quotienting by the relation $g^2 = 1$ as before) we can still apply our algebraic superization theory (21).

Recall that $A(R)$ is the bialgebra generated by 1 and $\mathbf{t} = \{t^i{}_j\}$ $i,j = 1,\ldots,d$ with algebra and coalgebra relations given by $R\,\mathbf{t}_1\,\mathbf{t}_2 = \mathbf{t}_2\,\mathbf{t}_1\,R$ and $\triangle\mathbf{t} = \mathbf{t}\otimes\mathbf{t}$, $\varepsilon(\mathbf{t}) = \mathrm{id}$, respectively. In particular, for the AC solution (1) we have:

Definition IV.1 *$A(R)$ is the matrix bialgebra generated by $\mathbf{t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and relations*

$$b\,a = q\,a\,b, \qquad c\,a = q\,a\,c, \qquad d\,b = -q^{-1}\,b\,d, \qquad d\,c = -q^{-1}\,c\,d,$$
$$c\,b = b\,c, \qquad d\,a - a\,d = (q - q^{-1})\,b\,c, \qquad b^2 = c^2 = 0, \tag{32}$$
$$\triangle\mathbf{t} = \mathbf{t}\otimes\mathbf{t}, \qquad \varepsilon(\mathbf{t}) = \mathrm{id}.$$

Assuming that a and d are invertible it is then possible to define the antipode by

$$S(\mathbf{t}) = \begin{pmatrix} a^{-1} + a^{-1}bd^{-1}ca^{-1} & -a^{-1}bd^{-1} \\ -d^{-1}ca^{-1} & d^{-1} + d^{-1}ca^{-1}bd^{-1} \end{pmatrix}, \tag{33}$$

which makes the bialgebra $A(R)$ into a Hopf algebra.

Unlike some other quantum groups, $SL(2)_q$ for example, the existence of each generator antipode does not require any quantum determinant condition. Furthermore, $S(\mathbf{t})$ can be computed much in analogy with supermatrices as follows: the antipode map axioms demand that $\mathbf{t}\cdot S(\mathbf{t}) = S(\mathbf{t})\cdot\mathbf{t} = 1$ so in matrix terms we must find a matrix $S(\mathbf{t})$ such that $S(\mathbf{t}) = \mathbf{t}^{-1}$. To obtain it we first split $\mathbf{t}$ in the sum of matrices $\mathbf{t}_0 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $\mathbf{t}_1 = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ and operate on $\mathbf{t}^{-1} = \left(\mathbf{t}_0\left(1 + \mathbf{t}_0^{-1}\mathbf{t}_1\right)\right)^{-1} = \left(1 - \mathbf{t}_0^{-1}\mathbf{t}_1 + \left(\mathbf{t}_0^{-1}\mathbf{t}_1\right)^2\right)\mathbf{t}_0^{-1}$, where the power serie truncates because of relations $b^2 = c^2 = 0$. The only assumption for this to be correct is that a, d are invertible elements.

Regarding the existence of a quantum determinant in the Hopf algebra there does not seem to be any group-like central element that could play the role of quantum determinant properly. We explain this in more detail noting that in $A(R)$ with a, d invertible the element

$$\mathcal{D}(\mathbf{t}) = ad^{-1} - bd^{-1}cd^{-1}$$

commutes with a, d and anticommutes with b, c, is invertible and group-like. It also satisfies the relation $\mathcal{D}(\mathbf{t}\cdot\mathbf{t}') = \mathcal{D}(\mathbf{t})\cdot\mathcal{D}(\mathbf{t}')$ for $\mathbf{t}$ and $\mathbf{t}'$ any two commuting quantum matrices whose elements satisfy (32). Of course, this result means that $\mathcal{D}^2$ is central and group-like so that we could think of it as the quantum determinant of $\mathbf{t}$. However, we refrain from calling it the quantum determinant since it is not particularly natural in this role. For instance, the relation $\mathcal{D}^2 = 1$ is not compatible with the duality pairing $\langle \mathbf{l}^{\pm}, \mathbf{t}\rangle = R^{\pm}$ with $U_q(K_1, K_2, X^{\pm})$ in the case of solution (1) since the value of $\mathcal{D}^2$ in the fundamental $(+)$ and conjugate fundamental $(-)$ representation of $A(R)$ given by

$$\rho^{+i}{}_j\,(t^k{}_l) = R^k{}_l{}^i{}_j, \qquad \rho^{-i}{}_j\,(t^k{}_l) = R^{-1}\,{}^i{}_j{}^k{}_l \tag{34}$$

is $\rho^{\pm}(\mathcal{D}^2) = q^{\pm 2}$. This indicates that we could naturally set either $\mathcal{D}^2 = q^2$ or $\mathcal{D}^2 = q^{-2}$ not to the unit. The derivation of this relation from R is according to [7], Sec 3.3.4.

These remarks about this non-standard quantum groups appear strange from the point of view of quantum group theory but, once again, become clear from the point of view of the corresponding super-quantum group obtained by superization. To do this we apply the transmutation theory of Section III by first extending it by the group algebra of $\boldsymbol{Z}_2$ as a Hopf algebra semidirect product $A(R){>\!\!\!\triangleleft}\boldsymbol{Z}_2$ and then applying Proposition III.1. This extension is nothing but $A(R)$ with generators a, b, c, d entirely unchanged and the extra generator g of $\boldsymbol{Z}_2$ adjoined, with $g^2 = 1$, $\triangle g = g \otimes g$ and the cross relations $a\,g = g\,a$, $b\,g = -g\,b$, $c\,g = -g\,c$, $d\,g = g\,d$. The product $A(R){>\!\!\!\triangleleft}\boldsymbol{Z}_2$ is still a Hopf algebra because the comultiplications of $A(R)$ and $\boldsymbol{Z}_2$ extend as an algebra homomorphism to $A(R){>\!\!\!\triangleleft}\boldsymbol{Z}_2$. In analogy with Proposition III.2 we have

Proposition IV.2 *Let $A(R)$ be the matrix quantum group (32) associated to the Alexander-Conway solution of the QYBE. The superization of its $\boldsymbol{Z}_2$-extension $A(R){>\!\!\!\triangleleft}\boldsymbol{Z}_2$ is isomorphic to the super $\boldsymbol{Z}_2$-extension $GL_q(1|1){>\!\!\!\triangleleft}\boldsymbol{Z}_2$. Here $GL_q(1|1)$ is the super-Hopf algebra with generators $\mathbf{u} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, α and δ even, β and γ odd and relations [10] [17]*

$$\beta\,\alpha = q\,\alpha\,\beta, \qquad \gamma\,\alpha = q\,\alpha\,\gamma, \qquad \delta\,\beta = q^{-1}\,\beta\,\delta, \qquad \delta\,\gamma = q^{-1}\,\gamma\,\delta,$$

$$\gamma\,\beta = -\beta\,\gamma, \qquad \delta\,\alpha - \alpha\,\delta = (q - q^{-1})\,\beta\,\gamma \qquad \beta^2 = \gamma^2 = 0,$$

$$\underline{\triangle}\mathbf{u} = \mathbf{u} \otimes \mathbf{u}, \qquad \underline{\varepsilon}(\mathbf{u}) = \mathrm{id}$$

and

$$\underline{S}(\mathbf{u}) = \begin{pmatrix} \alpha^{-1} + \alpha^{-1}\beta\delta^{-1}\gamma\alpha^{-1} & -\alpha^{-1}\beta\delta^{-1} \\ -\delta^{-1}\gamma\alpha^{-1} & \delta^{-1} + \delta^{-1}\gamma\alpha^{-1}\beta\delta^{-1} \end{pmatrix}. \tag{35}$$

Its $\boldsymbol{Z}_2$-extension is by adjoining a bosonic element g implementing the degree operator.

Proof From Section III we know that the superization of $A(R){>\!\!\!\triangleleft}\boldsymbol{Z}_2$ has the same algebra but a modified comultiplication. Redefining the generators of this Hopf algebra as

$$\alpha = a, \qquad \beta = b\,g, \qquad \gamma = c, \qquad \delta = d\,g. \tag{36}$$

is immediate to see that in the new generators the algebra is that of $GL_q(1|1){>\!\!\!\triangleleft}\,\boldsymbol{Z}_2$ while the superized comultiplication becomes the matrix one of $GL_q(1|1)$ and $\underline{\triangle}g = g\otimes g$ is unchanged. In $GL_q(1|1){>\!\!\!\triangleleft}\,\boldsymbol{Z}_2$ the cross relations with g are simply according to the super-statistics, i.e. g commutes with the even generators of $GL_q(1|1)$ and anticommutes with the odd ones. $\square$

We see in particular that the even combination $\mathcal{D}\,g = \alpha\delta^{-1} - \beta\delta^{-1}\gamma\delta^{-1}$, group-like and central in $A(R){>\!\!\!\triangleleft}\,\boldsymbol{Z}_2$, is viewed after superization as the usual super-quantum determinant of $GL_q(1|1)$ as in [17].

We conclude this section with a general theorem of which the above is an example. Its proof also supplies details of the proof of Proposition IV.2 previously sketched.

Theorem IV.3 *Let $R\in M_d\otimes M_d$ be a superizable matrix solution of the QYBE as in Definition III.3 and let $A(R)$ be the matrix FRT bialgebra associated to R. Let $A(R){>\!\!\!\triangleleft}\,\boldsymbol{Z}_2$ be the $\boldsymbol{Z}_2$-extension of $A(R)$ by adjoining an element g with $g^2=1$, $\triangle g = g\otimes g$ and the cross relations*

$$g\, t^i{}_j = (-1)^{p(i)\,+\,p(j)} t^i{}_j\, g.$$

Then $A(R){>\!\!\!\triangleleft}\,\boldsymbol{Z}_2$ is an ordinary bialgebra and its superization is isomorphic to $A(\underline{R}){>\!\!\!\triangleleft}\,\boldsymbol{Z}_2$, that is, to the $\boldsymbol{Z}_2$-extension of the super FRT bialgebra associated to $\underline{R}$.

Proof (i) First we check that we can make the $\boldsymbol{Z}_2$-extension of $A(R)$ as claimed. To do this we define an action of $\boldsymbol{Z}_2$ on $A(R)$ by $g\triangleright t^i{}_j = (-1)^{p(i)\,+\,p(j)}\, t^i{}_j$ extended to all $A(R)$ as a module algebra, i.e. $g\triangleright(t^i{}_j\, t^k{}_l) = (g\triangleright t^i{}_j)\,(g\triangleright t^k{}_l)$. For this to be consistent with the algebra relations $R\,\mathbf{t}_1\,\mathbf{t}_2 = \mathbf{t}_2\,\mathbf{t}_1\,R$ of $A(R)$ we need $(-1)^{p(f)\,+\,p(c)\,+\,p(e)\,+\,p(d)}\, R^a{}_f{}^b{}_e\, t^f{}_c\, t^e{}_d = (-1)^{p(b)\,+\,p(r)\,+\,p(a)\,+\,p(s)}\, t^b{}_r\, t^a{}_s\, R^s{}_c{}^r{}_d$. Since R is superizable, the extra factors are both $(-1)^{p(a)\,+\,p(b)\,+\,p(c)\,+\,p(d)}$ so they cancel. Hence the action of g extends to $A(R)$ and respects its algebra structure. It also respects the coalgebra structure because $(g\otimes g)\triangleright\triangle t^i{}_j = \sum_k (-1)^{p(i)\,+\,p(k)\,+\,p(k)\,+\,p(j)}\, t^i{}_k\otimes t^k{}_j = \triangle(g\triangleright t^i{}_j)$ as required. Since the action of g thus respects the algebra and coalgebra structure, it is immediate that the semidirect product $A(R){>\!\!\!\triangleleft}\,\boldsymbol{Z}_2$ by this action is a bialgebra.

(ii) The super FRT bialgebra $A(\underline{R})$ with generators 1, $\mathbf{u}=(u^i{}_j)$ of degree $p(u^i{}_j)=p(i)+p(j)$ is defined by relations

$$(-1)^{p(a)p(b)+p(c)p(e)}\,\underline{R}^a{}_f{}^b{}_e\,u^f{}_c\,u^e{}_d=(-1)^{p(c)p(d)+p(r)p(a)}\,u^b{}_r\,u^a{}_s\,\underline{R}^s{}_c{}^r{}_d, \tag{37}$$

and $\underline{\Delta}\mathbf{u}=\mathbf{u}\underline{\otimes}\mathbf{u}$, $\varepsilon(\mathbf{u})=\mathrm{id}$. Its $\mathbb{Z}_2$-extension $A(\underline{R})\rtimes\mathbb{Z}_2$ as a super-bialgebra consists of adjoining a bosonic element g with $g^2=1$, $\underline{\Delta}g=g\underline{\otimes}g$ and cross relations $g\,u^i{}_j=(-1)^{p(i)+p(j)}u^i{}_j\,g$. We show that $\theta: A(R)\rtimes\mathbb{Z}_2\to A(\underline{R})\rtimes\mathbb{Z}_2$ defined by $\theta(t^i{}_j)=u^i{}_j\,g^{p(j)}$ and $\theta(g)=g$ is well defined as an algebra isomorphism (notice here that transformation (36) is precisely of this type). Indeed, applying θ to both sides of $R\,\mathbf{t}_1\,\mathbf{t}_2=\mathbf{t}_2\,\mathbf{t}_1\,R$ we find that $(-1)^{p(c)(p(e)+p(d))}\,R^a{}_f{}^b{}_e\,u^f{}_c\,u^e{}_d$ $g^{p(c)+p(d)}=(-1)^{p(r)(p(a)+p(s))}\,u^b{}_r\,u^a{}_s\,R^s{}_c{}^r{}_d\,g^{p(r)+p(s)}$. Since R is superizable we can replace $g^{p(r)+p(s)}$ on the right by $g^{p(c)+p(d)}$ and hence cancel it. Writing now R in terms of $\underline{R}$ with (30) we obtain exactly the algebra relations (37) of $A(\underline{R})$. The cross relations also coincide, those in $A(\underline{R})\rtimes\mathbb{Z}_2$ being given by commutativity or anticommutativity according to the grading $p(u^i{}_j)=p(i)+p(j)$. Hence θ is an algebra map. The superization theorem applied to $A(R)\rtimes\mathbb{Z}_2$ does not change the algebra structure, but it does change the comultiplication. The new comultiplication from (21) is $\underline{\Delta}t^i{}_j=\sum_k t^i{}_k\,g^{p(k)+p(j)}\otimes t^k{}_j$ and $\underline{\Delta}g=g\otimes g$ (unchanged). From this it follows that $(\theta\otimes\theta)\underline{\Delta}(t^i{}_j\,g^{-p(j)})=\sum_k u^i{}_k\otimes u^k{}_j=\underline{\Delta}\circ\theta(t^i{}_j\,g^{-p(j)})$. Thus, after superizing $A(R)\rtimes\mathbb{Z}_2$, the map θ becomes an isomorphism of super bialgebras. This completes the proof of the theorem. Finally we remark that if $A(R)\rtimes\mathbb{Z}_2$ (or quantum group version of it) has an antipode map S then this is also superized and θ induces an antipode $\underline{S}$ on the corresponding version of $A(\underline{R})\rtimes\mathbb{Z}_2$ obeying $\underline{S}(g)=g$ and $\sum_k\underline{S}(u^i{}_k)\,u^k{}_j=\delta^i{}_j=\sum_k u^i{}_k\,\underline{S}(u^k{}_j)$. This is obtained by applying θ. $\square$

This theorem precisely connects the algebraic superization theory from [14] as used in Section III with the idea of replacing R by $\underline{R}$ and making a 'parrallel' super-FRT construction. The algebraic method works more generally and applies also to quotients of GL versions of $A(R)$ provided the additional relations are compatible with the $\mathbb{Z}_2$ action. We have given here the most easily explained setting for superization in which the (cross product) algebra does not change but the

coalgebra is made super. One can also view the passage from $A(R)$ to $A(\underline{R})$ as a change of product induced by a dual transmutation theory where p provides a map $A(R) \to \mathbb{Z}'_2$, as explained in [30, Appendix]. From this point of view $A(\underline{R}) = B(R, Z)$ where Z is a super-transposition R-matrix defined by p and $B(\ ,\ )$ is the more general transmutation construction recently studied more (as quantum braided groups) in [31] and elsewhere. See [30, Appendix] for details.

Let us note finally that there are many non-standard R-matrices beyond the specific solution (1) on which we focused, and to which the above superization constructions apply equally well. Thus the enveloping algebra super-quantum group $U_q gl(n|m)$ and the matrix super-quantum group $GL(n|m)_q$ are the superization of certain non-standard quantum groups. These are associated by super-FRT type constructions to the super R-matrices

$$\underline{R}_{gl(n|m)} = q\sum_{i=1}^{n} E_{ii}\otimes E_{ii} + q^{-1}\sum_{i=n+1}^{n+m} E_{ii}\otimes E_{ii} + \sum_{i\neq j} E_{ii}\otimes E_{jj} + (q-q^{-1})\sum_{j>i}(-1)^{p(i)p(j)}\, E_{ij}\otimes E_{ji}.$$

where E_{ij} here, for $1 \le i,\, j \le n+m$, are the $(n+m)\times(n+m)$-matrix given by $(E_{ij})^k{}_l = \delta^k{}_i \delta^l{}_j$ and $p(i)$ is the function with values 0,1 depending on whether $i = 1, \ldots n$ or $i = n+1, \ldots n+m$, respectively. This super R-matrix is the superization of the nonstandard solution of the ordinary QYBE

$$R_{gl(n|m)} = q\sum_{i=1}^{n} E_{ii}\otimes E_{ii} - q^{-1}\sum_{i=n+1}^{n+m} E_{ii}\otimes E_{ii} + \sum_{i\neq j}(-1)^{p(i)p(j)}\, E_{ii}\otimes E_{jj} + (q-q^{-1})\sum_{j>i} E_{ij}\otimes E_{ji},$$

of which (1) of previous sections constitutes the particular case $n = m = 1$. It leads by an FRT-type construction to bosonic (not super) non-standard quantum groups which are, however, strictly connected by superization with $U_q gl(n|m)$ and $GL_q(n|m)$. In particular, $R_{gl(n|m)}$ do not directly generate these super-quantum groups, as sometimes stated in the literature.

V q-supersymmetry of q-exterior algebras

In this section we develop a particular application of (a twisted version of) the non-standard quantum enveloping algebra $U_q(H_1, H_2, X^{\pm})$ studied in Section II. We consider the quantum exterior

algebra $\Omega_q(R)$ on a quantum plane of any dimension n, defined by generators $\left\{\begin{pmatrix} x_1\cdots x_n \\ \mathrm{d}x_1\cdots \mathrm{d}x_n\end{pmatrix}\right\}$ and relations

$$x_i\, x_j = q^{-1}\, x_b\, x_a R^a{}_i{}^b{}_j, \qquad \mathrm{d}x_i\, x_j = q\, x_b\, \mathrm{d}x_a R^a{}_i{}^b{}_j, \qquad \mathrm{d}x_i\, \mathrm{d}x_j = -q\, \mathrm{d}x_b\, \mathrm{d}x_a R^a{}_i{}^b{}_j, \tag{38}$$

where R is any $n^2\times n^2$ matrix solution of the Yang-Baxter equations obeying the Hecke condition $(P\,R-q)(P\,R+q^{-1}) = 0$. Here P denotes the permutation matrix. The Hecke condition is sufficient to ensure that relations (38) are compatible with the usual graded Leibniz rule and $\mathrm{d}^2=0$. Exterior algebras of this (and more complicated) form are quite well-studied now [18] [19]. We have grouped the generators, however, as a rectangular quantum matrix $A(R_\Omega : R)$ [32], which are defined like the usual FRT relations but with respect to two R-matrices. As explained in [30], this means automatically that there is a braided addition law on $\Omega_q(R)$ (i.e., one does not need to show this, as some authors have done) and, relevant to us, a coaction from the left of the quantum group $A(R_\Omega)$. Here

$$R_\Omega = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & q-q^{-1} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & -q^{-1}\end{pmatrix} \tag{39}$$

is a close relative of (1) and defines the non-standard matrix quantum group $A(R_\Omega)$ with generators 1, $\mathbf{t} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ and relations [30]

$$b\,a = a\,b, \qquad c\,a = q^2\, a\,c, \qquad d\,b = -b\,d, \qquad d\,c = -q^{-2}\, c\,d,$$
$$c\,b = q^2 b\,c, \qquad d\,a - a\,d = -(1-q^2)\,b\,c, \qquad b^2 = c^2 = 0.$$

Then $\Omega_q(R)$ is covariant under matrix multiplication from the left by such quantum matrices (as well as being covariant from the right under $A(R)$, which is the usual point of view).

This non-standard quantum group $A(R_\Omega)$ is different from, but a close relative of, the non-standard quantum group studied in Section IV. Our first result makes this precise at the dual level of the 'enveloping' algebras.

Proposition V.1 *The QYBE solution* R_Ω *and the ansatz*

$$\mathbf{l}^+ = \begin{pmatrix} q^{(H_1-H_2)/2} & 0 \\ (q-q^{-1})\, q^{H_1/2}\, X^+ & e^{-i\frac{\pi}{2}H_2}\, q^{(H_1-H_2)/2} \end{pmatrix}, \quad \mathbf{l}^- = \begin{pmatrix} q^{-(H_1+H_2)/2} & -(q-q^{-1})\, q^{-H_2/2}\, X^- \\ 0 & e^{i\frac{\pi}{2}H_2}\, q^{(H_1+H_2)/2} \end{pmatrix}$$

leads to the quantum enveloping algebra $U_q^\Omega(H_1, H_2, X^\pm)$ *with the same algebra and counit as in (7) in Section II but the coproduct and antipode*

$$\triangle_\Omega H_i = H_i \otimes 1 + 1 \otimes H_i,$$

$$\triangle_\Omega X^+ = X^+ \otimes q^{-H_2/2} + e^{-i\frac{\pi}{2}H_2}\, q^{-H_2/2} \otimes X^+, \quad \triangle_\Omega X^- = X^- \otimes e^{i\frac{\pi}{2}H_2}\, q^{(H_1+2\,H_2)/2} + q^{-H_1/2} \otimes X^-,$$

$$S_\Omega\,(H_i) = -H_i, \qquad S_\Omega\,(X^+) = -q\, e^{i\frac{\pi}{2}H_2}\, q^{H_2}\, X^+, \qquad S_\Omega\,(X^-) = q\, e^{-i\frac{\pi}{2}H_2}\, q^{-H_2}\, X^-.$$

This quantum group $U_q^\Omega(H_1, H_2, X^\pm)$ *is a twisting in the sense of [20] [21] of the nonstandard quantum group* $U_q(H_1, H_2, X^\pm)$ *of Section II, but with the quantum cocycle*

$$\chi = q^{\dfrac{1}{4}\, H_2 \otimes (H_1 + H_2)}$$

Proof We chose the ansatz so that the algebra relations of $U(\widetilde{R_\Omega})$ recover the same relations (7) as in Section II. The matrix coproduct of $\mathbf{l}^\pm$ then determines the coproduct of $H_i, X^\pm$, which comes out differently from Section II. We can recognize it as of the twisted form $\triangle_\Omega h = \chi(\triangle h)\chi^{-1}$ for all h in $U_q(H_1, H_2, X^\pm)$. We check finally that χ here itself obeys the quantum 2-cocycle condition $(1 \otimes \chi)(\mathrm{id} \otimes \triangle)\chi = (\chi \otimes 1)(\triangle \otimes \mathrm{id})\chi$ and $(\epsilon \otimes \mathrm{id})\chi = 1$ as required for the twisting theory. $\square$

So this non-standard quantum group, while not exactly the one in Section II, is 'gauge equivalent' to it in the sense of twisting. This means, for example, that we have at once its universal R-matrix as $\mathcal{R}_\Omega = \chi_{21}\, \mathcal{R}\, \chi^{-1}$ in terms of the one found in Section II, namely

$$\mathcal{R}_\Omega = e^{-i\dfrac{\pi}{4} H_2 \otimes H_2}\; q^{\dfrac{1}{4}\left((H_1 - H_2) \otimes (H_1 + H_2)\right)} \Big(1 \otimes 1 + (1 - q^2)\, E \otimes q^{-(H_1+H_2)/2}\, F\Big)$$

for the quantum group $U_q^\Omega(H_1, H_2, X^\pm)$.

Proposition V.2 *The non-standard quantum enveloping algebra* $U_q^{\Omega}(H_1, H_2, X^{\pm})$ *acts covariantly from the right on any quantum exterior algebra* $\Omega_q(R)$ *associated to a Hecke solution* R *of the QYBE. The action is*

$$x_i{\triangleleft} H_1 = 2\,x_i, \quad x_i{\triangleleft} H_2 = 0, \quad x_i{\triangleleft} X^+ = q^{-1}\,\mathrm{d}x_i, \quad x_i{\triangleleft} X^- = 0,$$

$$\mathrm{d}x_i{\triangleleft} H_1 = 0, \quad \mathrm{d}x_i{\triangleleft} H_2 = 2\,\mathrm{d}x_i, \quad \mathrm{d}x_i{\triangleleft} X^+ = 0, \quad \mathrm{d}x_i{\triangleleft} X^- = q\,x_i.$$

Proof The left coation of $A(R_\Omega)$ dualizes to a right action of $U_q^{\Omega}(H_1, H_2, X^{\pm})$. We compute it by evaluating the matrix coaction against the generators $\mathbf{l}^{\pm}$ in the usual way, which in our case means in terms of the R-matrix R_Ω. This then gives the form stated for the action of our $H_1, H_2, X^{\pm}$ generators. Since the coaction is an algebra homomorphism it means, as one can check explicitly, that the action is covariant in the sense $(x\,y){\triangleleft} h = (x{\triangleleft} h_{(1)})(y{\triangleleft} h_{(2)})$ for all $x, y \in \Omega_q(R)$ and $\triangle_\Omega h = h_{(1)}{\otimes}h_{(2)}$, say. $\square$

So this non-standard quantum group has a geometrical role as a hidden quantum group symmetry (valid even for $q = 1$) of the exterior algebra of a quantum plane, mixing x_i and $\mathrm{d}x_i$.

Finally, superization allows us to pass systematically to a super version of these results.

Proposition V.3 *The superization of* $A(R_\Omega)$ *is the matrix super-bialgebra* $A(\underline{R}_\Omega)$ *with generators 1 and* $\mathbf{u} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, α *and* δ *even,* β *and* γ *odd and relations*

$$\beta\,\alpha = \alpha\,\beta, \qquad \gamma\,\alpha = q^2\,\alpha\,\gamma, \qquad \delta\,\beta = \beta\,\delta, \qquad \delta\,\gamma = q^{-2}\,\gamma\,\delta,$$

$$\gamma\,\beta = -q^2\,\beta\,\gamma, \qquad \delta\,\alpha - \alpha\,\delta = (1 - q^2)\,\beta\,\gamma \qquad \beta^2 = \gamma^2 = 0$$

and matrix super coproduct. If we assume α *and* δ *invertible, we obtain a super-quantum group* $GL_q^{\Omega}(1|1)$, *say, with antipode given by the usual formula (35). The super determinant is again a central, bosonic and group-like element given by*

$$\underline{\det}(\mathbf{u}) = \alpha\delta^{-1} - \beta\delta^{-1}\gamma\delta^{-1}.$$

Proof The matrix R_Ω is superizable with $p(1)=0$, $p(2)=1$. We compute $\underline{R}_\Omega$ from (30) (only a sign changes) and put it into (37). If we allow a,d invertible we can obtain a nonstandard quantum group with antipode, much as in Proposition IV.2. Superising this gives a superantipode if we assume α,δ invertible. $\square$

Next, a general feature of the transmutation theory is that representations and covariant systems under the original object remain so (but in the new category) under the transmuted one. Hence in particular, representations or covariant systems under the original bosonic quantum group become automatically (in our case) super-representations or super-covariant systems under the superization. We check this explicitly for our example.

Proposition V.4 *Let $\Omega_q(R)$ be a quantum exterior algebra of Hecke type, regarded as a super algebra with x_i even and $\mathrm{d}x_i$ odd. Then (keeping in mind the corresponding bose-fermi statistics) it is covariant under the $GL_q^\Omega(1|1)$ transformation*

$$\begin{pmatrix} x'_1\cdots x'_n \\ \mathrm{d}x'_1\cdots \mathrm{d}x'_n\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix} x_1\cdots x_n \\ \mathrm{d}x_1\cdots \mathrm{d}x_n\end{pmatrix}$$

in the sense that the primed co-ordinates and forms obey the same relations of $\Omega_q(R)$.

Proof We verify directly that indeed $\mathrm{d}x'_i\,\mathrm{d}x'_j = -q\,\mathrm{d}x'_b\,\mathrm{d}x'_a\,R^a{}_i{}^b{}_j$. From the stated super-transformation we have that $\mathrm{d}x'_i\,\mathrm{d}x'_j + q\,\mathrm{d}x'_b\,\mathrm{d}x'_a\,R^a{}_i{}^b{}_j = (\gamma\,x_i+\delta\,\mathrm{d}x_i)\,(\gamma\,x_j+\delta\,\mathrm{d}x_j) + q\,(\gamma\,x_b+\delta\,\mathrm{d}x_b)\,(\gamma\,x_a+\delta\,\mathrm{d}x_a)\,R^a{}_i{}^b{}_j = \gamma\,\delta\,x_i\mathrm{d}x_j - \delta\,\gamma\,\mathrm{d}x_i\,x_j + q\,\gamma\,\delta\,x_k\mathrm{d}x_l\,R^l{}_i{}^k{}_j - q\,\delta\,\gamma\,\mathrm{d}x_k\,x_l\,R^l{}_i{}^k{}_j = \gamma\,\delta\,x_i\mathrm{d}x_j - \delta\,\gamma\mathrm{d}x_i\,x_j + \gamma\,\delta\,\mathrm{d}x_i\,x_j - q\,\delta\,\gamma\,x_e\mathrm{d}x_f R^f{}_k{}^e{}_l\,R^l{}_i{}^k{}_j = 0$. In the last equality we used the Hecke condition in the form $R^f{}_k{}^e{}_l\,R^l{}_i{}^k{}_j = (q-q^{-1})\,R^f{}_i{}^e{}_j + \delta^f_j\,\delta^e_i$. Similarly for the other relations of $\Omega_q(R)$. $\square$

The supersymmetry in this form appears to be related to a somewhat larger 'universal super-bialgebra' coacting on exterior algebras, developed by other means in [24].

Finally, to complete our picture, we construct the corresponding superenveloping algebra $U_q^\Omega gl(1|1)$ either by algebraic superization as in Section III or from $\underline{R}_\Omega$ and the ansatz

$$\mathbf{m}^+ = \begin{pmatrix} q^{h-2N} & 0 \\ (q-q^{-1})\, q^{h-N}\,\eta & q^{h-2N} \end{pmatrix}, \qquad \mathbf{m}^- = \begin{pmatrix} q^{-h} & -(q-q^{-1})\, q^{-N}\,\eta^+ \\ 0 & q^h \end{pmatrix}.$$

Proposition V.5 *The super-quantum group $U_q^\Omega gl(1|1)$ has generators h, N, η, η^+ obeying the same algebra relations as $U_q gl\,(1|1)$ but different supercomultiplication, superantipode and super-universal R-matrix*

$$\begin{gathered} \underline{\triangle}_\Omega h = h\otimes 1 + 1\otimes h, \qquad \underline{\triangle}_\Omega N = N\otimes 1 + 1\otimes N \\ \underline{\triangle}_\Omega \eta = \eta\otimes q^{-N} + q^{-N}\otimes\eta, \qquad \underline{\triangle}_\Omega \eta^+ = \eta^+\otimes q^{h+N} + q^{-h+N}\otimes\eta^+ \end{gathered} \tag{40}$$

$$\underline{S}_\Omega(h) = -h, \qquad \underline{S}_\Omega(N) = -N, \qquad \underline{S}_\Omega(\eta) = -q\,q^{2N}\,\eta, \qquad \underline{S}_\Omega(\eta^+) = -q\,q^{-2N}\,\eta^+, \tag{41}$$

$$\underline{\mathcal{R}}_\Omega = q^{-2\,N\,\otimes\,h}\left(1\otimes 1 + (1-q^2)\,q^N\,\eta\otimes q^{-N-h}\,\eta^+\right). \tag{42}$$

It is related by twisting of super-quantum groups to $U_q gl(1|1)$ via $\underline{\chi} = q^{N\otimes h}$, viewed as a super quantum 2-cocycle. Moreover, it acts covariantly on any $\Omega_q(R)$ of Hecke type by

$$x_i\triangleleft h = x_i, \quad x_i\triangleleft N = 0, \quad x_i\triangleleft \eta = q^{-1}\,\mathrm{d}x_i, \quad x_i\triangleleft\eta^+ = 0$$

$$\mathrm{d}x_i\triangleleft h = \mathrm{d}x_i, \quad \mathrm{d}x_i\triangleleft N = \mathrm{d}x_i, \quad \mathrm{d}x_i\triangleleft\eta = 0, \quad \mathrm{d}x_i\triangleleft\eta^+ = q\,x_i$$

Proof The relations of $U(\widetilde{\underline{R}_\Omega})$ for the ansatz shown leads to the usual relations of $U_q gl(1|1)$. The matrix supercoproduct of the $\mathbf{m}^\pm$ gives the form on the generators. We recognize it as $\underline{\triangle}$ in (24) twisted as $\underline{\triangle}_\Omega(\cdot) = \underline{\chi}\,\underline{\triangle}(\cdot)\,\underline{\chi}^{-1}$ for $\underline{\chi}$ as shown. It is as super 2-cocycle in the sense

$$(1\otimes\underline{\chi})\,(\mathrm{id}\otimes\underline{\triangle})\,\underline{\chi} = (\underline{\chi}\otimes 1)\,(\underline{\triangle}\otimes\mathrm{id})\,\underline{\chi}$$

and $(\underline{\varepsilon}\otimes\mathrm{id})\,\underline{\chi} = 1$. This then gives at once the super universal R-matrix as stated, obtained from (25) as $\underline{\mathcal{R}}_\Omega = \underline{\chi}_{21}\,\underline{\mathcal{R}}\,\underline{\chi}^{-1}$. Note that, in the present case, $\underline{\chi}$ is bosonic so it is an ordinary 2-cocycle just as well; the bosonic twisting (as in Proposition V.1) followed by superisation gives the same

answer as superising first and then twisting as we do now. Finally, we dualize the coaction in the preceding proposition via the super duality relations

$$\begin{aligned} &< u^i{}_j,\, m^{+\,k}{}_l >= (-1)^{p(j)(p(k)+p(l))}\, \underline{R_\Omega}{}^i{}_j{}^k{}_l, \\ &< u^i{}_j,\, m^{-\,k}{}_l >= (-1)^{p(k)(p(i)+p(j))}\, \underline{R_\Omega}{}^{-1\,k}{}_l{}^i{}_j \end{aligned} \tag{43}$$

to obtain the right actions as stated. It follows (as one can also verify directly) that this right action of $U_q^\Omega gl(1|1)$ is super-covariant in the sense

$$(x\,y){\triangleleft}\,h = (-1)^{\deg(y)\ \deg(h_{\underline{(1)}})}\,(x{\triangleleft}\,h_{\underline{(1)}})\,(y{\triangleleft}\,h_{\underline{(2)}}) \tag{44}$$

for all $x, y \in \Omega_q(R)$ and $\underline{\triangle}_\Omega h = h_{\underline{(1)}} \otimes h_{\underline{(2)}}$ the super-coproduct. The action on products of generators of $\Omega_q(R)$ is consistently determined by this super-covariance. $\square$

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